Chapter 2

Basic equations

In this chapter, the basic equations are considered that describe velocity, pressure, temperature and composition in viscous fluid models as applied to the Earth's interior.

2.1 Conservation equations

Neglecting inertial forces, the equations of conservation of mass, momentum and thermal energy can be written for a Boussinesq fluid in the field of gravity as

$$\nabla \cdot \mathbf{u} = 0 \tag{2.1}$$

$$\nabla p = \nabla \cdot \underline{\sigma} + \rho g \hat{\mathbf{z}} \tag{2.2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T + Q/\rho c_p \tag{2.3}$$

where the meaning of the symbols is specified in table 2.1. For multicomponent fluids conservation of composition (assuming negligible mass diffusion) yields

$$\frac{\partial \Gamma}{\partial t} + (\mathbf{u} \cdot \nabla)\Gamma = 0 \tag{2.4}$$

Together with the boundary and/or initial conditions and the equation of state

$$\rho = \rho(T, \Gamma) \tag{2.5}$$

these equations describe the motion of the fluid, and mass and heat transport for flows driven by thermal and chemical buoyancy forces in the field of gravity.

2.2 Rheological model

The deformation behaviour of mantle rock at high temperatures is described, here, by power law ductile creep [e.g. Kirby, 1983], where strain rate is related to the *n*th power of the deviatoric stress

$$\dot{\varepsilon}_{ij} = A\sigma^{n-1}\sigma_{ij} \tag{2.6}$$

Here σ_{ij} are the components of the deviatoric stress tensor $\underline{\sigma}$, $\dot{\varepsilon}_{ij}$ the components of the strain rate tensor $\underline{\dot{\varepsilon}}$ and σ is the second invariant of the deviatoric stress tensor, or the effective shear stress

$$\sigma = \left[\frac{1}{2} \sum_{ij} \sigma_{ij}\sigma_{ij} \right]^{\frac{1}{2}} \tag{2.7}$$

Equivalently, the second invariant of the strain rate tensor, or the effective strain rate, is defined by

$$\dot{\varepsilon} = \left[\frac{1}{2} \sum_{ij} \dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij} \right]^{\frac{1}{2}} \tag{2.8}$$

The components of the strain rate tensor are given by

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \tag{2.9a}$$

An effective isotropic shear viscosity η is then given by,

$$\eta = \frac{\sigma}{2\dot{\varepsilon}} \tag{2.9b}$$

In engineering and fluid dynamics literature often a slightly different definition of the strain rate tensor and viscosity is used:

$$\dot{\varepsilon}_{ij} = \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] \tag{2.10a}$$

$$\eta = \frac{\sigma}{\dot{\varepsilon}} \tag{2.10b}$$

For these two sets of definitions the factor A in (2.6) has a different meaning. This should be taken into account when comparing different models or when applying laboratory measurements of strain rate in mathematical models.

For computational purposes the stress dependence of viscosity is expressed in terms of strain rate, through an inversion of (2.6), giving

$$\eta = \frac{\sigma}{\dot{\varepsilon}} = A^{-1/n} \cdot \dot{\varepsilon}^{\frac{1-n}{n}} \tag{2.11}$$

2.3 Thermal convection

In convection driven by thermal buoyancy forces alone, the equation of state $\rho = \rho(T)$ is written to a first order approximation as

$$\rho = \rho_0 + \delta\rho = \rho_0 - \alpha\rho_0(T - T_0) \tag{2.12}$$

where $\rho_0 = \rho$ at temperature $T = T_0$ and α the thermal expansion coefficient. Elimination of hydrostatic pressure from (2.2) leads to

$$-\nabla P + \nabla \cdot (\eta \dot{\underline{\varepsilon}}) = \alpha\rho_0 g(T - T_0)\hat{\mathbf{z}} \tag{2.13}$$

where $P = p - \rho_0 gz$ is the hydrodynamic pressure.
The equations can be non-dimensionalized using the scaling parameters given in table 2.1. Introducing primed quantities $X' = rX$, where r is the scaling parameter, we find for (2.1-2.3)

$$-\nabla P + \nabla \cdot (\eta \dot{\underline{\varepsilon}}) = RaT\hat{\mathbf{z}} \tag{2.14}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.15}$$

$$\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla)T = \nabla^2 T + Q \tag{2.16}$$

where primes have been dropped and

$$Ra = \frac{\rho_0 g\alpha\Delta T_0 h^3}{\kappa_0\eta_0} \tag{2.17}$$

is the (non-dimensional) thermal Rayleigh number.
The three equations (2.14-2.15) can be reduced to one scalar equation through introduction of the stream function ψ, defined by $\mathbf{u} = (\partial\psi/\partial y\,,\ -\partial\psi/\partial x)^{\mathrm{T}}$, where we choose the y-axis in the negative z-direction. In non-dimensional quantities, the depth is given by $z = 1 - y$. Use of the stream function yields, in dimensionless form

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right]\eta\left[\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right] + 4\frac{\partial^2}{\partial x\partial y}\eta\frac{\partial^2\psi}{\partial x\partial y} = Ra\frac{\partial T}{\partial x} \tag{2.18}$$

The continuity equation is now automatically satisfied, at the cost of fourth order derivatives in the resulting differential equation.

Table 2.1 Specification of symbols and (if applicable) non-dimensionalisation factors (after [Christensen, 1984a]). R is the gas constant ($R = 8.314\ \mathrm{Jmol^{-1}K^{-1}}$), g the acceleration of gravity ($g = 9.8\ \mathrm{m \cdot s^{-2}}$), h a specific length scale (in general the depth of the layer), c_p specific heat at constant pressure, and κ_0, η_0, ΔT_0 arbitrarily chosen reference values. The temperature scale is chosen, such that $T_0 = 0$.

Symbol	Meaning	Unit	factor r for non-dimensionalisation
x, y	horizontal, vertical coordinate	m	$1/h$
$\mathbf{u}$	velocity vector, $\mathbf{u} = (v, w)^T$	$\mathrm{m \cdot s^{-1}}$	h/κ_0
∇	$\nabla = (\partial/\partial x, \partial/\partial y)^T$	$\mathrm{m^{-1}}$	h
$\hat{\mathbf{z}}$	unit vector in direction of gravity		
t	time	s	κ_0/h^2
ψ	stream function	$\mathrm{m^2s^{-1}}$	$1/\kappa_0$
η	dynamical viscosity	$\mathrm{Pa \cdot s}$	$1/\eta_0$
$\underline{\sigma}$	deviatoric stress tensor	Pa	$h^2/(\kappa_0\eta_0)$
$\dot{\underline{\varepsilon}}$	strain rate tensor	$\mathrm{s^{-1}}$	h^2/κ_0
ρ	mass density	$\mathrm{kg \cdot m^{-3}}$	$1/\rho_0$
κ	thermal diffusivity	$\mathrm{m^2s^{-1}}$	$1/\kappa_0$
α	thermal expansion coefficient	$\mathrm{K^{-1}}$	
T	temperature	K	$1/\Delta T_0$
p	pressure	Pa	$h^2/(\kappa_0\eta_0)$
Γ	composition function		
Q	Internal heating per unit volume	$\mathrm{Wm^{-3}}$	$h^2/(c_p\kappa_0\Delta T_0\rho_0)$

2.4 Thermochemical convection

When both temperature and composition determine the density, the equation of state becomes

$$\rho = \rho(\Gamma)[1 - \alpha(T - T_0)] \tag{2.19}$$

For a model with two layers of different composition, Γ can be defined as a step function and the compositional density is then given by

$$\rho(\Gamma) = \rho_0 + \Delta\rho\Gamma \tag{2.20}$$

where $\Delta\rho$ is the density difference between the two layers. Similar formulations are possible for multi-layer fluids or models with continuously varying composition. For this particular two-layer case we can write

$$\frac{\partial\rho}{\partial x} = \Delta\rho\,\frac{\partial\Gamma}{\partial x} - [\rho_0 + \Gamma\Delta\rho]\alpha\,\frac{\partial T}{\partial x} \tag{2.21}$$

and the equation of motion in the stream function formulation becomes

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right]\eta\left[\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right] + 4\,\frac{\partial^2}{\partial x\partial y}\,\eta\,\frac{\partial^2\psi}{\partial x\partial y} = \left[\frac{\rho_0 + \Gamma\Delta\rho}{\rho_0}\right]Ra\,\frac{\partial T}{\partial x} - Rb\,\frac{\partial\Gamma}{\partial x} \tag{2.22}$$

where Rb is the so-called boundary Rayleigh number

$$Rb = \frac{\Delta\rho g h^3}{\kappa_0\eta_0} \tag{2.23}$$